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A TITS ALTERNATIVE FOR GROUPS THAT ARE RESIDUALLY OF BOUNDED RANK

BY

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ABSTRACT

In this short note a Tits alternative for certain kinds of groups which are residually of rank at most r is obtained. The main theorem states that if G is a group that is residually (locally (soluble-by-finite) of rank r), then either G is locally (soluble-by-finite) or G contains a non-abelian free subgroup.

1. Introduction

Let r be a fixed positive integer. A group G has (Prüfer) rank r if every finitely generated subgroup of G can be generated by r elements and r is the least such integer. Throughout this paper we shall say that a group G has rank r if, in the above sense, it has rank at most r; no confusion should arise as a result. In an earlier paper [3] we discussed the class of groups which are residually of rank r for some fixed natural number r and in particular we showed that a group which is residually (of rank r and locally soluble) is locally nilpotent-by-residually linear-by soluble. We denote the class of groups which are residually (of rank r) by res

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(rank r). More generally, if $\mathcal{P}$ is a class (or property) of groups and the group G is residually a $\mathcal{P}$-group we shall say that G is a res ($\mathcal{P}$) group. We recall that a group is said to be almost $\mathcal{P}$ if it has a $\mathcal{P}$-subgroup of finite index. In his seminal paper [7], J. Tits proved that a finitely generated linear group either is almost soluble or contains a non-abelian free subgroup. This important result underlies most of our work here.

In this short note we continue the investigation of groups which are res (rank r) and obtain a "Tits alternative" for various kinds of groups with this property. We shall prove the following theorem:

THEOREM A: *Let G be a res (locally (soluble-by-finite) of rank r) group. Then either G is locally (soluble-by-finite) or G contains a non-abelian free subgroup.*

In general, one cannot conclude that G is locally (soluble-by-finite). For let F be an arbitrary non-abelian free group and p a prime. Then clearly F is res (free and finitely generated) and it follows from [3, Theorem 6] that F is res (finite p-group of rank 9).

Clearly a locally (soluble-by-finite) group need not be almost locally soluble. However, N.S. Černikov [1] has shown that a locally (soluble-by finite) group of finite rank is almost locally soluble. Thus Theorem A is really about groups which are res (almost locally soluble and of rank r). Theorem A should perhaps be compared with 3.1 of [3].

Let G be a direct product of infinitely many copies of a fixed non-abelian finite simple group of rank r; thus G is res (finite of rank r) and has no non-abelian free subgroups. Since G is not almost locally soluble, it follows that we cannot replace "locally (soluble-by-finite)" with "almost locally soluble" in the conclusion of Theorem A. Following [3], let us say that a group G is res*($\mathcal{P}$) if it has a countable descending series of normal subgroups N_i, intersecting in the identity, and such that each G/N_i is a $\mathcal{P}$-group. Our next result is an immediate consequence of Theorem A and [3, Theorem 7].

COROLLARY A: *Let G be a res* (locally (soluble-by-finite) of rank r) group. Then either G is almost locally soluble or G contains a non-abelian free subgroup.*

As above, we may use [3, Theorem 6] to show that each free group F is res*(finite p and of rank 9) for each prime p.

Our other main result, Theorem C below, provides a more specific Tits alternative for finitely generated res (soluble of rank r) groups. This extends a result of Segal [6]. Its proof depends heavily on ideas from [6]. These allow us to establish a result about modules over finitely generated soluble groups which is of interest in its own right. We say that a module A is **res (rank r) as a module** if there exists a collection $\{A_i\}_{i\in I}$ of submodules of A such that $\bigcap_{i\in I} A_i = 0$ and A/A_i is an abelian group of rank r.

THEOREM B: *Let G be a finitely generated soluble group and let A be a $\mathbb{Z}G$-module that is res (rank r) as a module. Then there exists an integer k and a subgroup H of finite index in G such that $[A,_k H'] = 0$.*

Here $[A,_k H']$ denotes the group $[A, H', H', \ldots, H']$ where there are k occurrences of H'.

THEOREM C: *Let G be a finitely generated res (soluble of rank r) group. Then G has an abelian-by-nilpotent normal subgroup Q such that G/Q is a subdirect product of finitely many linear groups. If G contains no non-abelian free subgroups then G is nilpotent-by-abelian-by-finite.*

2. The proofs of the Theorems

In order to prove Theorem A we require the following consequence of the classification of finite simple groups.

LEMMA 1: *Let H be a finite semisimple group of rank r. Then there is an integer m, depending only on r, such that H is res (linear of degree m).*

Proof: We first prove the following

CLAIM: *Let Ω be the class of all non-abelian finite simple groups of rank r. Then there exists an integer N' such that $\mathrm{Aut}\, S$ is linear of degree N' for each $S \in \Omega$.*

Clearly $\mathrm{Out}\, S$ has bounded rank as S runs through the finite simple alternating groups and the finitely many sporadic groups. Moreover, $\mathrm{Out}\, S$ has rank 5 if S is of Lie type (see, for example, [2, page xvi]). Thus there exists $z \in \mathbb{N}$ such that $\mathrm{Aut}\, S$ has rank z for each $S \in \Omega$. In particular, if $S \in \Omega$ then $\mathrm{Aut}\, S$ has no section of type $C_p \wr C_n$ with p prime and $n > z$. The claim now follows from [9, 4.1].

Now let $\mathrm{soc}\, H$ denote the socle of H. It is well-known that

$$\mathrm{soc}\, H = M_1 \times M_2 \times \cdots \times M_t$$

for some $t \le r$ where M_i is a direct product of $k_i \le r$ copies of a non-abelian finite simple group S_i and $S_i \ncong S_j$ if $i \ne j$. Here, of course, the bounds on t and k_i are consequences of the Feit–Thompson theorem. Note that S_i has rank r for $i = 1, \ldots, t$. Now H embeds in $\mathrm{Aut}(M_1 \times \cdots \times M_t) \cong \mathrm{Aut}\, M_1 \times \cdots \times \mathrm{Aut}\, M_t$ and

$\mathrm{Aut}\, M_i \cong (\mathrm{Aut}\, S_i) \wr \Sigma_{k_i}$ where the wreath product is with respect to the natural permutation action of Σ_{k_i}, the symmetric group of degree k_i. Since $\mathrm{Aut}\, S_i$ is linear of degree N', it is easy to see that $\mathrm{Aut}\, M_i$ is linear of degree $k_i(N')(k_i!)$. The result now follows easily. ∎

The proof of our next result is very easy and is left to the reader.

LEMMA 2: *A locally (soluble-by-finite) group that is residually soluble is locally soluble.*

We need one other preliminary result before we prove Theorem A.

LEMMA 3: *Let G be res (locally soluble of rank r). Then either G is locally soluble or G contains a non-abelian free subgroup.*

Proof: Suppose that G contains no non-abelian free subgroups. By Theorem 2 of [3], there are subgroups M, N of G with $M \lhd N \lhd G$ such that M is locally nilpotent, N/M is residually (linear of r-bounded degree) and G/N is soluble (of r-bounded derived length). If H is a finitely generated subgroup of N, then [9, 4.2] and our assumption on G together imply that $H/H \cap M$ is almost soluble. However Theorem 4 of [3] then implies that H is almost soluble. Since H is residually soluble, Lemma 2 now shows that H is soluble and hence N is locally soluble. By [3, Theorem 4] again it follows that N is radical, whence so is G, and a final application of [3, Theorem 4] now gives the result. ∎

Proof of Theorem A: Let G be as stated and suppose G contains no non-abelian free subgroups. As we mentioned above, a result of Černikov [1] shows that G is actually res (almost locally soluble and of rank r). Thus there exists a collection $\{N_i\}_{i \in I}$ of normal subgroups of G, indexed by some set I, such that $\bigcap_{i \in I} N_i = 1$ and G/N_i is almost locally soluble, for each $i \in I$. Let R_i/N_i be the locally soluble radical of G/N_i and $R = \bigcap_{i \in I} R_i$. By considering the collection $\{R \cap N_i\}_{i \in I}$ of normal subgoups of R, it is easy to see that R is res (locally soluble of rank r) and hence, by Lemma 3, R is locally soluble. Now G/R is res (finite semisimple of rank r) and so Lemma 1 implies that there exists an integer m such that G/R is res (linear of degree m). Moreover, since G/R has no non-abelian free subgroups, we may apply [9, 4.2] and deduce that G/R is locally (soluble-by-finite). Thus, if H is a finitely generated subgroup of G, we have that $H/(H \cap R)$ is soluble-by-finite and $H \cap R$ is locally soluble. Since H is res (rank r), it follows easily from [3, Theorem 4] that H is soluble-by-finite. The proof is complete. ∎

If X is an abelian group we shall let $T(X)$ denote the torsion subgroup of X.

Proof of Theorem B: Using the notation introduced before the statement of Theorem B we may, on modifying the index set if necessary, assume that $T(A/A_i)$ is a p_i-group for some prime p_i. Let $B_i = A/A_i, T_i = T(B_i)$ and let D_i be the divisible radical of T_i. Let

$$B = \operatorname*{Cr}_{i\in I} B_i, \quad T = \operatorname*{Cr}_{i\in I} T_i \quad \text{and} \quad D = \operatorname*{Cr}_{i\in I} D_i.$$

Now B is a $\mathbb{Z}G$-module in a natural way and A embeds in B. We shall show that for some integer k, there is a subgroup H of finite index in G such that $[B,_k H'] = 0$. The result will then follow since A is a $\mathbb{Z}G$-submodule of B.

Since B_i/T_i is a torsionfree abelian group of rank r we have that $\operatorname{Aut}(B_i/T_i)$ embeds in $\mathrm{GL}(r, \mathbb{Q})$. Now a theorem of Mal'cev [4, Theorem 3.21] shows that there exists $q = q(r)$ such that every soluble linear group of degree r has a unipotent-by-abelian subgroup of index at most q. Since G is finitely generated it has only finitely many subgroups of a given finite index and hence if L is the intersection of all the subgroups of G of index at most q then $|G : L| < \infty$ and L' acts unipotently on B_i/T_i. Hence, for all i, we have $[(B_i/T_i),_r L'] = 0$. It follows that

$$[B/T,_r L'] = 0. \tag{1}$$

Now consider D. It is well-known that $\operatorname{Aut} D_i \leq \mathrm{GL}(r, \mathbb{Z}_{p_i})$ where $\mathbb{Z}_p$ denotes the ring of p-adic integers so that, as above, we have

$$[D,_r L'] = 0. \tag{2}$$

The remainder of the proof is very similar to the proof of the main theorem of [6]. Let

$$R = \operatorname*{Cr}_{i\in I} \mathbb{Z},$$

a commutative ring. For each i, T_i/D_i is a finite abelian p_i-group of rank r, so $C = T/D$ is r-generator as an R-module. Moreover, G acts on C by R-module automorphisms. Since G is finitely generated, there exist a finitely generated subring S of R and an r-generator S-submodule M of C such that $MG = M$ and $MR = C$. As in [6], it now follows that G has a subgroup F of finite index such that $M(F' - 1)^s = 0$ for some $s \in \mathbb{N}$. Since $MR = C$ it follows that $C(F' - 1)^s = 0$. Thus

$$[T/D,_s F'] = 0. \tag{3}$$

We set $H = F \cap L$. Then H has finite index in G and (1), (2) and (3) together imply that $[B,_{2r+s} H'] = 0$ and the result follows. ∎

In order to prove Theorem C we require the following special case.

LEMMA 4: *Let G be a finitely generated soluble group which is residually of rank r. Then G is nilpotent-by-abelian-by-finite.*

Proof: We prove this result by induction on the derived length d of G. If $d = 1$ the result is clear, so assume that $d > 1$ and that the result is true for all finitely generated soluble groups which are res (rank r) and of derived length at most $d-1$. Let A be a maximal normal abelian subgroup of G containing $G^{(d-1)}$. Clearly G/A is finitely generated and soluble of derived length at most $d-1$. It is also easy to show that G/A is res (rank r), by the choice of A. Hence, by induction, G/A is nilpotent-by-abelian-by-finite and to prove the result we may assume that G/A is nilpotent-by-abelian. Now A is a $\mathbb{Z}G$-module and is res (rank r) as a module. By Theorem B, G has a subgroup H of finite index such that $[A,_k H'] = 1$, for some integer k. Since $H'A/A$ is nilpotent it follows that $H'A$ is nilpotent and G is nilpotent-by-abelian-by-finite as required. ∎

Finally we prove Theorem C. Our original proof depended on Theorem A and therefore on the classification of finite simple groups. We are grateful to the referee for the following proof which does not depend on the classification.

Proof of Theorem C: Let G be as stated. First note that if H is a finitely generated soluble group of finite rank then the finite residual of H is abelian (see [4, 10.38 and 10.33]). Let R denote the (finite-soluble) residual of G and let $x, y \in R$. Let $M \lhd G$ be such that G/M is soluble of finite rank and observe that xM and yM are in the finite residual of G/M. Thus $[x, y] \in M$ and we deduce that R is abelian. The theorem of [6] shows that G/R contains a normal nilpotent subgroup Q/R such that G/Q is a subdirect product of finitely many linear groups and the first part of the theorem follows.

Suppose now that G contains no non-abelian free subgroups. Then G/Q contains no non-abelian free subgroups and it follows easily from Tits's theorem [7] that G/Q is soluble-by-finite. Thus G is soluble-by-finite and Lemma 2 shows that G is soluble. The result is now an immediate consequence of Lemma 4. ∎

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